Journal of Statistical Physics, Vol. 78, Nos. 5/6, 1995

The Chiral Potts Model and Its Associated Link Invariant

F. Y. Wu,[1] P. Pant,[1] and C. King[2]

Received June 20, 1994

A new link invariant is derived using the exactly solvable chiral Potts model and a generalized Gaussian summation identity. Starting from a general formulation of link invariants using edge-interaction spin models, we establish the uniqueness of the invariant for self-dual models. We next apply the formulation to the self-dual chiral Potts model, and obtain a link invariant in the form of a lattice sum defined by a matrix associated with the link diagram. A generalized Gaussian summation identity is then used to carry out this lattice sum, enabling us to cast the invariant into a tractable form. The resulting expression for the link invariant is characterized by roots of unity and does not appear to belong to the usual quantum group family of invariants. A table of invariants for links with up to eight crossings is given.

KEY WORDS: Link invariants; chiral Potts model; generalized Gaussian summation identity.

1. INTRODUCTION

Knots and links are embeddings of circles in $\mathbf{R}^3$ described by their projections onto a plane. As projections change their configurations when the embedded circles are deformed in $\mathbf{R}^3$, it is pertinent to ask what is being preserved in the process of deformation. Obviously, what is being preserved is intrinsic to the topology of the link, and this leads to the consideration of link invariants.

Link invariants are algebraic quantities associated with planar projections, which remain unchanged when the links are deformed. An exciting recent development in the theory of knots is the realization that link invariants can be obtained from exactly solvable models in statistical

[1] Department of Physics, Northeastern University, Boston, Massachusetts 02115.
[2] Department of Mathematics, Northeastern University, Boston, Massachusetts 02115.

0022-4715/95/0300-1253$07.50/0 © 1995 Plenum Publishing Corporation

mechanics (for reviews of this development see refs. 1–3). Indeed, using solvable models in two dimensions, it has been possible to generate all known link invariants of the quantum group family, including the Jones[4] and the Homfly[5] polynomials. More recent developments on spin models and link invariants can be found in refs. 6–11.

In a recent letter[12] we reported a new link invariant derived from the solvable chiral Potts model[13, 14] and evaluated using a generalized Gaussian summation identity.[15] The new invariant, which was earlier defined by Kobayashi *et al.*[16] without explicit evaluations, is in the form of a polynomial of roots of unity. In view of its novelty and the intimate relation with the Gaussian summation identity, it seems useful to provide details of our analysis. This is the purpose of the present paper. In addition, we also present a general duality consideration, a discussion of some properties and relations of our invariant to other known polynomial invariants, and a table of the invariant for links with eight or fewer crossings.

This paper is organized in the following manner. In Section 2 we recall the formulation of link invariants using edge-interaction models, paying particular attention to models with chiral interactions. In Section 3 we present the formulation of a duality relation for general edge-interaction spin models and show that it leaves unchanged the invariant derived from self-dual models. In Section 4 we recall the solvable chiral Potts model, and a certain infinite-rapidity limit of its vertex weights is introduced in Section 5. Section 6 deals with the detailed evaluation of invariants. We derive the Skein relation satisfied by the invariant and discuss relations with other known invariants in Section 7. Several technical points and some properties of the invariant are discussed in Section 8. In particular, we establish the identity of the invariant with its mutant, a result we extend to any invariant derivable from edge-interaction spin model. In Appendix A we describe the summation identities known as the Gaussian summation formula. A table of invariants for all links with up to eight crossings is given in Appendix B.

2. LINK INVARIANTS FROM EDGE-INTERACTION MODELS

Link invariants can be generated from solvable two-dimensional models in statistical mechanics.[1-3] Here, we briefly review the formulation involving edge-interaction models.

Starting from a link K which we assume to be oriented, one constructs a directed lattice $\mathscr{L}$ by regarding lines of the link as lattice edges and line crossings as lattice vertices. This leads naturally to two types of vertices, $+$ and $-$, corresponding to the two kinds of line crossings $+$ and $-$ in the

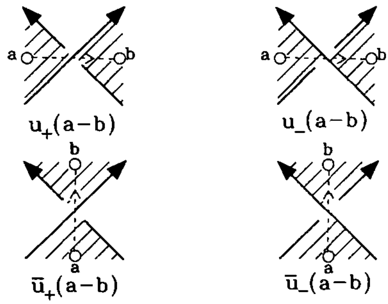

Fig. 1. The + and − line intersections associated with two kinds of shadings that can occur at a vertex and the vertex weights.

link as shown in Fig. 1. Now the lines divide faces of $\mathscr{L}$ into two subsets, with one subset neighboring only faces of the other subset, and vice versa. This permits us to introduce a spin model with spins residing in one subset of the faces and interactions extending across the line intersections. The spins and the interactions form a line graph G. To help one to visualize, it is customary to shade the faces in which the spins reside.[17] Then, the line graphs for the two different shadings are mutually dual. The example of the two different face shadings for the link 8^2_{15} is shown in Fig. 2. Note that there exist four distinct types of line crossings and, consequently, four types of spin interactions. These four different configurations, shown in Fig. 1, possess weights

$$u_\pm(a-b), \qquad \bar{u}_\pm(a-b) \tag{1}$$

where $a, b = 1, 2,..., N$ are the spin states of the two interacting spins. Here, we have assumed quite generally that the interactions can be chiral, namely, $u_\pm(a)$ can be different from $u_\pm(-a)$. The example of the two line graphs for the link 8^2_{15} is shown in Fig. 2 with the type of interactions explicitly noted.

Let $Z(u_\pm, \bar{u}_\pm)$ be the partition function of the spin model for a given face shading. Then, the formulation of link invariants using spin models[1,3] dictates that $Z(u_\pm, \bar{u}_\pm)$ is a link invariant, provided that the Boltzmann weights satisfy certain conditions imposed by Reidemeister moves.[18] Reidemeister moves are elementary moves of lines in the knot projection when links are deformed in $\mathbf{R}^3$. The possible Reidemeister moves that can occur are those shown in Fig. 3, where for each line movement we must

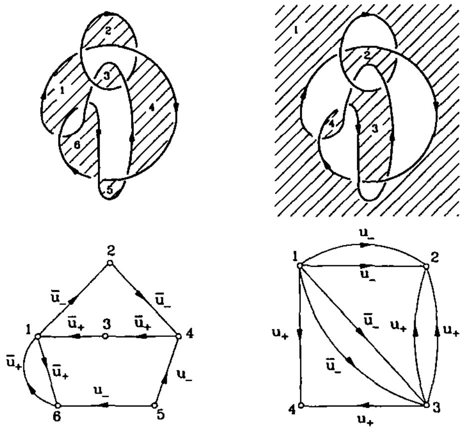

Fig. 2. The two different kinds of face shadings and the associated line graphs for the link 8^2_{15}, where the interaction types are explicitly noted.

allow the two possible kinds of face shadings. The desired Reidemeister conditions can be read off from the figure, leading to

$$\bar{u}_\pm(0) = 1 \tag{2a}$$

$$\frac{1}{\sqrt{N}} \sum_{b=0}^{N-1} u_\pm(a-b) = 1 \tag{2b}$$

$$u_+(a-b)\, u_-(a-b) = 1 \tag{2c}$$

$$\frac{1}{N} \sum_{b=0}^{N-1} \bar{u}_+(a-b)\, \bar{u}_-(b-c) = \delta_{ac} \tag{2d}$$

$$\bar{u}_+(a-b)\, \bar{u}_-(b-a) = 1 \tag{2e}$$

$$\frac{1}{N} \sum_{b=0}^{N-1} u_+(a-b)\, u_-(c-b) = \delta_{ac} \tag{2f}$$

$$\frac{1}{\sqrt{N}} \sum_{d=0}^{N-1} u_-(a-d)\, \bar{u}_-(b-d)\, \bar{u}_+(d-c) = \bar{u}_-(b-c)\, u_-(a-c)\, u_+(a-b) \tag{2g}$$

Provided that conditions (2a)–(2g) are met, the quantity

$$I_K(N) \equiv N^{(c_D - S - 2)/2} Z(u_\pm, \bar{u}_\pm) \tag{3}$$

where S is the number of spins (shaded faces) in $\mathscr{L}$ and c_D is the number of connected components of the line graph associated with the other (dual)

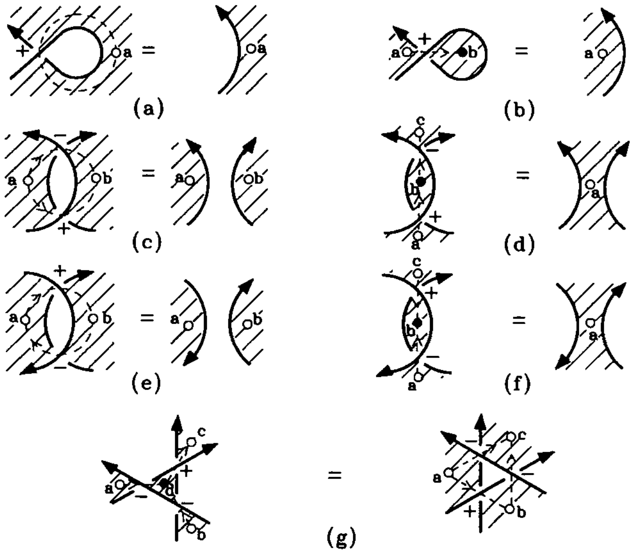

Fig. 3. Reidemeister moves for oriented knots with two different kinds of face shadings.

choice of face shading, is an invariant for the link K.[3,12] Note that $c_D = 1$ for connected $\mathscr{L}$. Note also that the normalization of (3) has been taken to be $I_{\text{unknot}}(N) = 1$.

3. DUALITY RELATION FOR SPIN MODELS

In this section we present the formulation of a general duality relation for two-dimensional edge-interaction spin models in a form suitable for our consideration.

Consider an N-state spin model on a line graph G with S sites. The spins interact with a generally chiral interaction $u_{ij}(a-b)$ along the lattice edge $\{i, j\}$ connecting sites i and j in respective spin states a and b. Consider the dual model whose spins are on the dual graph G_D, with S_D sites and with interactions

$$u^{(D)}(n) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \omega^{mn} u(m) \tag{4}$$

whose inverse is

$$u(n) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} \omega^{-mn} u^{(D)}(m) \tag{5}$$

where $\omega = e^{2\pi i/N}$.

Let c and c_D be the respective numbers of connected components of G and G_D. (For connected lattices we have $c = c_D = 1$.) Let $Z(\{u\})$ and $Z^{(D)}[\{u^{(D)}\}]$ be the respective partition functions of the spin models on G and G^D. Then, following Wu and Wang,[19] one establishes the identity

$$Z(\{u\}) = N^{c - S_D} Z^{(D)}[\{\sqrt{N}\, u^{(D)}\}] \tag{6}$$

where the partition function on the right-hand side of (6) is defined with interactions $\sqrt{N}\, u^{(D)}$ in place of $u^{(D)}$. Using the Euler relation generalized to disjoint graphs

$$S + S_D = c + c_D + E \tag{7}$$

where E is the number of edges (which is the same for G and G_D), one obtains

$$N^{(c_D - S - 2)/2} Z(\{u\}) = N^{(c - S_D - 2)/2} Z^{(D)}[\{u^{(D)}\}] \tag{8}$$

This is the desired duality relation.

For chiral interactions for which lattice edges are directed, the orientation of a lattice edge on G_D is that of the corresponding lattice edge on G rotated 90°.[19] Furthermore, the shading of the other set of faces, namely, choosing spins to reside in the other set of faces, corresponds to the interchange of $u_\pm \leftrightarrow \bar{u}_\pm$ (cf. Fig. 1). Therefore, a $\bar{u}$ interaction on G corresponds in the dual space to a $u_\pm^{(D)}$ interaction, and a u interaction on G corresponds on G_D to a $\bar{u}_\pm^{(D)}$. This leads to the duality relation

$$N^{(c_D - S - 2)/2} Z(u_\pm, \bar{u}_\pm) = N^{(c - S_D - 2)/2} Z^{(D)}[\bar{u}_\pm^{(D)}, u_\pm^{(D)}] \tag{9}$$

It follows that for self-dual models satisfying $\bar{u}_\pm^{(D)} = u_\pm$, $u_\pm^{(D)} = \bar{u}_\pm$, the invariant (3) evaluated using either scheme of face shading is identically the same. This conclusion applies to both the chiral and nonchiral self-dual models.

4. THE CHIRAL POTTS MODEL

The N-state chiral Potts model is a spin model whose Boltzmann weights $W(n)$ and $\bar{W}(n)$ are chiral and N-periodic. Namely, quite generally we have $W(-n) \neq W(n)$ and $\bar{W}(-n) \neq \bar{W}(n)$ and the equalities $W(n) = W(n+N)$ and $\bar{W}(n) = \bar{W}(n+N)$. An important recent advance in two-dimensional lattice statistics has been the discovery of an exact integrable manifold of the chiral Potts model.[13, 14] In the notation of ref. 14, the integrable vertex weights are best described by introducing

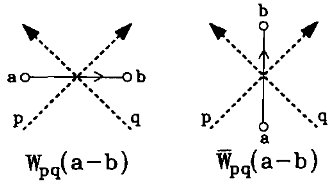

Fig. 4. Auxiliary lines defining rapidities for the chiral Potts model and the vertex weights.

rapidities a_p, b_p, c_p, d_p to auxiliary lines drawn as shown in Fig. 4. The rapidities satisfy the N-periodicity condition

$$\frac{a_p^N \pm b_p^N}{c_p^N \pm d_p^N} = \lambda_{\pm}, \qquad \text{independent of } p \tag{10}$$

Then the weights can be written, for $0 \leqslant n \leqslant N-1$, as

$$\begin{aligned} g_{pq}(n) &\equiv \frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \\ \bar{g}_{pq}(n) &\equiv \frac{\bar{W}_{pq}(n)}{\bar{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \end{aligned} \tag{11}$$

The weights (11) satisfy the relations

$$g_{pp}(n) = 1, \qquad \bar{g}_{pp}(n) = \delta_{n0} \tag{12a}$$

$$g_{pq}(0) = \bar{g}_{pq}(0) = 1 \tag{12b}$$

$$g_{pq}(a-b)\, g_{qp}(a-b) = 1 \tag{12c}$$

and the Yang–Baxter equation

$$\begin{aligned} &\sum_{d=0}^{N-1} W_{pr}(a-d)\, \bar{W}_{qr}(b-d)\, \bar{W}_{pq}(d-c) \\ &\qquad = R_{pqr}\, \bar{W}_{pr}(b-c)\, W_{qr}(a-c)\, W_{pq}(a-b) \end{aligned} \tag{12d}$$

where

$$\begin{aligned} R_{pqr} &= f_{pq} f_{qr} / f_{pr} \\ f_{pq} &= \left[\prod_{m=0}^{N-1} \frac{\sum_{k=1}^{N} \omega^{km} \bar{W}_{pq}(k)}{W_{pq}(m)} \right]^{1/N} \end{aligned} \tag{13}$$

For

$$c_p = d_p = 1$$
$$a_p^N + b_p^N = I = \text{const} \tag{14}$$

the model is self-dual[20] and f_{pq} is independent of p and q.

5. THE INFINITE-RAPIDITY LIMIT

To generate link invariants we need first to identify the weights $u_\pm$ and $\bar{u}_\pm$ appearing in (2a)–(2g). Comparing (2) with (12), one observes that an obvious choice is $\{u_+, u_-, \bar{u}_+, \bar{u}_-\} \sim \{g_{pq}, g_{qp}, \bar{g}_{pq}, \bar{g}_{qp}\}$. This leads us to introduce the infinite-rapidity limits

$$\begin{aligned} u_+(n) &= A_+ \lim_{b_p \to \infty} g_{pq}(n), \qquad & u_-(n) &= A_- \lim_{b_q \to \infty} g_{pq}(n) \\ \bar{u}_+(n) &= B_+ \lim_{b_p \to \infty} \bar{g}_{pq}(n), \qquad & \bar{u}_-(n) &= B_- \lim_{b_q \to \infty} \bar{g}_{pq}(n) \end{aligned} \tag{15}$$

where $A_\pm$ and $B_\pm$ are constants to be determined. It turns out that conditions (2a)–(2g) can be satisfied for the self-dual model (14) with $I = 0$. This leads us to write

$$a_p = \omega^{l - 1/2} b_p \tag{16}$$

where l is an integer, and

$$\begin{aligned} u_\pm(n) &= A_\pm (-1)^n\, \omega^{\pm(nl + n^2/2)} \\ \bar{u}_\pm(n) &= B_\pm (-1)^n\, \omega^{nl \mp n^2/2} \end{aligned} \tag{17}$$

While (17) is defined for $0 \leqslant n \leqslant N-1$, it follows from the N-periodic property that it holds for all n (positive or negative). Note that these weights are chiral for $l \neq 0$.

From (2a) and using the Gaussian summation identity (A2) with $M = 1$ in Appendix A to evaluate (2b), one obtains

$$\begin{aligned} A_\pm &= (-1)^l\, \omega^{\pm l^2/2}\, e^{\pm i\pi(N-1)/4} \\ B_\pm &= 1 \end{aligned} \tag{18}$$

It can then be verified that conditions (2c)–(2g) are now all satisfied by (17) and (18). It can also be verified from (4) that we have

$$\begin{aligned} u_\pm^{(D)}(n) &= \bar{u}_\pm(-n) \\ \bar{u}_\pm^{(D)}(n) &= u_\pm(n) \end{aligned} \tag{19}$$

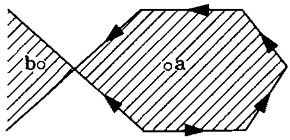

Fig. 5. Example of a shaded polygon showing one source and one sink of arrows around its perimeter.

Thus, as discussed at the end of Section 3, the invariant $I_K(N)$ is independent of the face shading chosen. This is the desired result.

The invariant $I_K(N)$ is also independent of l in (16). To see this, we observe from the second line of (17), which now holds for all n, that the $\bar{u}_{\pm}(a-b)$ interaction between spin states a and b contributes to the summand of the partition function an l-dependent factor $\omega^{la}\omega^{-lb}$. The two factors in this product can be split and associated separately to the two corners of the two shaded polygons meeting at the line crossing (cf. the second row of Fig. 1). Next we collect these factors for each polygon. Let the spin state of a shaded polygon be a. Then, to each corner of the polygon with two outgoing (incoming) arrows [source (sink) of arrows] we have a factor ω^{-la} (ω^{la}). As demonstrated in the example shown in Fig. 5, there is always an equal number of sources and sinks around a polygon. Thus, the l-dependent factors around each polygon cancel out and, as a result, the link invariant $I_K(N)$ does not change its value if we set $l=0$ in $\bar{u}_{\pm}$. As discussed in Section 7 below, every link has a special projection in which all crossings are of the type $\bar{u}_{\pm}$. It follows that $I_K(N)$ is independent of l. For self-dual models the dual of the $u_{\pm}$ weights are the $\bar{u}_{\pm}$ weights. Therefore, it also follows that we can set $l=0$ in $u_{\pm}$ (when evaluated in the dual space), and hence deduce directly that $I_K(N)$ is independent of l.

Finally, combining (17) with (18) and $l=0$, we arrive at the expressions

$$\begin{aligned} u_{\pm}(n) &= (-1)^n\, e^{\pm i(N-1)\pi/4}\, \omega^{\pm n^2/2} \\ \bar{u}_{\pm}(n) &= (-1)^n\, \omega^{\mp n^2/2} \end{aligned} \tag{20}$$

which are nonchiral. The substitution of (20) into (3) leads to the desired link invariants. It is interesting to observe that the chirality of the weights (17) is irrelevant in the resulting invariants.

6. EVALUATION OF KNOT INVARIANTS

The link invariant $I_K(N)$ for a link K is evaluated by substituting (20) into (3). Noting that the essential difference between the weights u and $\bar{u}$ in

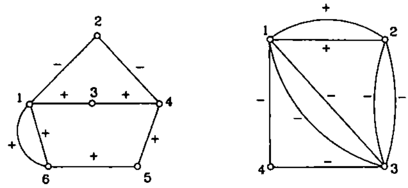

Fig. 6. Signed graphs correspond to the two possible face shadings of the link 8^2_{15} shown in Fig. 2.

(20) is the sign of the factor $n^2/2$ in the exponent, it is useful to note explicitly this sign, + or −, on the line graph, and this leads us to consider the signed graph associated with a face shading. Specifically, to each edge l in a line graph G, we assign a number $\varepsilon_l = \pm 1$ according to the following rules: if the associated line crossing is of type $\bar{u}_+$ or u_-, assign $\varepsilon_l = +1$; if the crossing is of type $\bar{u}_-$ or u_+, assign $\varepsilon_l = -1$. The example of the two signed graphs associated with the line graphs for the link 8^2_{15} of Fig. 1 is shown in Fig. 6. Note that this sign is determined by the face shading only, independent of the orientation of lines in $\mathscr{L}$. Thus, our choice of the sign agrees with that of Fig. 48 of ref. 3.

To facilitate bookkeeping, we now introduce an $S \times S$ (incidence) matrix $\mathbf{Q}$ with elements

$$\begin{aligned} Q_{ij} &= \sum_{l=\langle i,j\rangle} \varepsilon_l, \qquad i \neq j \\ Q_{ii} &= -\sum_{k(\neq i)} Q_{ik} \end{aligned} \tag{21}$$

where the summation in the first line is over *all* edges l connecting the ith and jth sites. The matrix $\mathbf{Q}$ is obviously symmetric; it also has the property that the sum of each row or column vanishes. Matrices possessing these properties are singular, all cofactors are equal, and the cofactors generate spanning trees of the graph G.[21]

Let $\mathbf{n} = (n_1, ..., n_S)$ be an integer-valued vector whose components $n_i = 0, 1, ..., N-1$, $i = 1, ..., S$, denote the state of the ith spin on G. Further introduce a vector $\mathbf{z}$ with components $z_i = Q_{ii}/2$, $i = 1, ..., S$. Then the link invariant (3), after the substitution with (20), assumes the form

$$\begin{aligned} I_K(N) = N^{(c_D - S - 2)/2} &\left[\exp \frac{\pi i(N-1)\,\zeta(K)}{4}\right] \\ &\times \sum_{n_1, ..., n_S = 0}^{N-1} \exp\left[\frac{\pi i}{N} \mathbf{n} \cdot (\mathbf{Q}\mathbf{n}) + 2\pi i \mathbf{n} \cdot \mathbf{z}\right] \end{aligned} \tag{22}$$

where $\zeta(K)$ is the number of u_+ weights minus the number of u_- weights in K.

Due to the N-periodicity of Boltzmann weights, there is no loss of generality to fix one of the S spins, say, the Sth, in the spin state $n_S=0$. Then the summation over n_S in (21) gives rise to a factor N and one obtains

$$I_K(N)=N^{(c_D-S)/2}\left[\exp\frac{\pi i(N-1)\,\zeta(K)}{4}\right] \times \sum_{n_1,\ldots,n_{S-1}=0}^{N-1} \exp\left[\frac{\pi i}{N}\mathbf{n}\cdot(\mathbf{Mn})+2\pi i\mathbf{n}\cdot\mathbf{y}\right] \tag{23}$$

where $\mathbf{n}$ and $\mathbf{y}$ are $(S-1)$-dimensional vectors, $\mathbf{M}$ is the $(S-1)\times(S-1)$ cofactor matrix of $\mathbf{Q}$ obtained by deleting the Sth row and column, and the summation is $(S-1)$-fold. Note that we have always

$$N(2y_i+M_{ii})=2NM_{ii}=\text{an even integer} \tag{24}$$

The expression (23) subject to (24) can be evaluated using a generalized Gaussian summation formula given in Appendix A. Provided that the matrix $\mathbf{M}$ is nonsingular, using (A5) we find

$$I_K(N)=N^{(c_D-1)/2}\left[\exp\frac{\pi i(N-1)\,\zeta(K)}{4}\exp\frac{\pi i\eta(\mathbf{M})}{4}\right] \times\frac{1}{\sqrt{D}}\sum_{\mathbf{n}\in\Delta}\exp\left\{-\pi iN(\mathbf{n}+\mathbf{y})\cdot[\mathbf{M}^{-1}(\mathbf{n}+\mathbf{y})]\right\} \tag{25}$$

where $\mathbf{n}=(n_1,\ldots,n_{S-1})$, $\mathbf{y}=(y_1,\ldots,y_{S-1})$, $y_i=M_{ii}/2$, $D=|\det\mathbf{M}|$, Δ is the fundamental domain (unit cell) of the lattice formed by the collection of vectors $\mathbf{Mn}$, and $\eta(\mathbf{M})$ is the signature of $\mathbf{M}$, namely, the number of positive eigenvalues minus the number of negative eigenvalues. Note that N now appears as a parameter, instead of a summation limit, in (25). The expression (25) completes the evaluation of $I_K(N)$.

It is instructive to illustrate the evaluation of (25) for the link 8^2_{15} shown in Fig. 1. The two signed graphs corresponding to the two different face shadings are shown in Fig. 6, from which one obtains the matrices

$$\mathbf{M}=\begin{pmatrix}-2&-1&1&0&0\\-1&2&0&-1&0\\1&0&-2&1&0\\0&-1&1&-1&1\\0&0&0&1&-2\end{pmatrix},\qquad\begin{pmatrix}1&2&-2\\2&0&-2\\-2&-2&5\end{pmatrix} \tag{26}$$

for the two shadings, respectively. In the first case one finds further $\zeta(K) = -2$, $\eta(\mathbf{M}) = -3$, $D = 8$, $\Delta = (-2, 1, 0, 0, -1)$, $(-1, 0, -1, 1, -1)$, $(-1, 0, 0, 0, 0)$, $(-1, 1, 0, 0, -1)$, $(-1, 1, -1, 0, 0)$, $(0, 0, -1, 1, -1)$, $(0, 0, 0, 0, 0)$, and $(0, 1, -1, 0, 0)$. In the second case $\mathbf{M}$ is a 3×3 matrix. One finds $\zeta(K) = 2$, $\eta(\mathbf{M}) = -3$, $D = 8$, $\Delta = (-2, -1, 2)$, $(-1, -1, 1)$, $(-1, 0, 0)$, $(-1, 0, 1)$, $(0, 0, -1)$, $(0, 0, 0)$, $(0, 1, -2)$, and $(1, 1, -3)$. In either case, (25) yields the invariant

$$I_K(N) = \frac{1}{\sqrt{2}} e^{-i\pi/4} \left(1 + e^{-7\pi i N/8} + e^{-3\pi i N/2} + e^{-15\pi i N/8}\right)$$

We have used a computer program to compute invariants from (25). Generally, for a given link K with a given orientation, the matrix $\mathbf{M}$ and the number $\zeta(K)$ can be read off from the link diagram and are used as inputs. The computer then searches for all sites in the fundamental domain Δ and evaluates (25) term by term. We include in Appendix B a table of invariants for all links with eight or fewer crossings. We note that our invariant assumes the same form for some links. Examples are the pairs $\{6_2, 7_2\}$, $\{6_3, 8_1\}$, $\{7_5, 8_2\}$, $\{8_6, 8_7\}$, $\{8_{10}, 8_{11}\}$, $\{8_{12}, 8_{13}\}$, and $\{7_7^2, w(K) = -1; 4_1^2, w(K) = 4\}$.

7. RELATIONS TO OTHER INVARIANTS AND SKEIN RELATIONS

The matrix $\mathbf{M}$ occurring in (22) has been used in previous studies of link invariants, in particular by Goeritz,[22] and has come to be known as the Goeritz matrix.[23] Traldi[24] introduced an extension of $\mathbf{M}$, which he called the modified Goeritz matrix, and this was the starting point for Kobayashi *et al.*,[16] who built their invariant $T_N(K)$ from this matrix. As alluded to in Section 1, $T_N(K)$ turns out to be identical to $I_K(N)$, although in a different form and without an explicit evaluation. The Goeritz matrix is also related to the Seifert matrix of the link. For completeness, we briefly define relevant notions and describe some related results.

Consider an oriented link diagram with a shading, where the infinite region is one of the shaded regions. Denote by $\alpha_0, \alpha_1, ..., \alpha_p$ these shaded regions, where $p = S - 1$ and α_0 is the infinite region. Denote by $\beta_1, ..., \beta_q$ the regions in the dual shading. The diagram is a *special projection* if the following conditions are satisfied:

(i) All β-regions have consistently oriented boundaries.

(ii) All β-regions are topological disks, without holes.

It is not difficult to show that every link has a special projection.[23] Since the link diagram is the projection of a link in space onto a plane, we may consider the β-regions in a special projection as the projection onto the plane of an oriented surface in space, whose boundary is the link. This surface is the Seifert surface of the link. It follows that $p=2g$, where g is the genus of the Seifert surface.

Suppose we have a special projection of a link. Place spins in the (shaded) α-regions as described in Section 2 and define the $\mathbf{Q}$ matrix associated with the projection as in Section 6. Omitting the infinite region α_0 reduces $\mathbf{Q}$ to the matrix $\mathbf{M}$. Since each α-region has an even number of boundary components, the diagonal entries of $\mathbf{M}$ are always even. Furthermore, all crossings are now of type $\bar{u}_{\pm}$, so $\zeta(K)=0$. Hence the invariant $I_K(N)$ given by (25) reduces to the simple form

$$I_K(N)=N^{(cD-1)/2}\left[\exp\frac{\pi i\eta(\mathbf{M})}{4}\right]\frac{1}{\sqrt{D}}\sum_{n\in\Delta}\exp(-\pi iN\mathbf{n}\cdot\mathbf{M}^{-1}\mathbf{n}) \tag{27}$$

In this situation $\eta(\mathbf{M})$ is itself a link invariant, and is called the *signature* of the link. As noted in ref. 12, $D=|\Delta(-1)|$, where $\Delta(t)$ is the Alexander polynomial. This fact can also be seen by considering a Seifert matrix as follows.

Let a_i be a simple closed curve on the Seifert surface which projects to a simple closed curve which encloses the region α_i, in the positive direction, but does not enclose any other α-regions. Let a_i^- be the "push-off," or a small displacement, of a_i from the surface in the direction opposite to the orientation of the surface. Then the Seifert matrix $\mathbf{S}$ is a $p\times p$ matrix whose (i,j)th element is the linking number of a_i^- and a_j for all $1\leqslant i,j\leqslant p$.[23] It is a straightforward calculation to show that $\mathbf{M}=\mathbf{S}+\mathbf{S}^T$, where $\mathbf{S}^T$ is the transpose of $\mathbf{S}$. It is also known that the Alexander polynomial is $\Delta(t)=\det(\mathbf{S}^T-t\mathbf{S})$ and that the signature of the link is that of the matrix $\mathbf{S}+\mathbf{S}^T$. These results imply $D=|\det\mathbf{M}|=|\Delta(-1)|$.

Since the matrix $\mathbf{M}$ in (23) can be expressed in terms of the Seifert matrix, it is natural to ask whether $I_K(N)$ contains different information than the Alexander polynomial $\Delta(t)$ and the signature invariant, which are also expressed in terms of the Seifert matrix. One example shows that the answer is yes. Let $\bar{K}$ denote the mirror image of K. Then, it can be verified that the four *nonequivalent* knots 6_1, $\overline{6_1}$, 9_{46}, and $\overline{9_{46}}$ share the same Alexander polynomial and signature. However, the invariant $I_K(N)$ distinguishes among 6_1, $\overline{6_1}$, and 9_{46}, although not between 9_{46} and $\overline{9_{46}}$. On the other hand, the Alexander polynomial distinguishes the links $\{8_{12}, 8_{13}\}$ for which $I_K(N)$ is identical.

Finally, the invariants $I_K(N)$ satisfy a Skein relation.[12] In particular, $I_K(2)$ and $I_K(3)$ are related to the Jones polynomial at special values. The Jones polynomial $V_K(t)$ is determined by the Skein relation

$$\frac{1}{t} V_{K_+}(t) - t V_{K_-}(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K_0}(t) \tag{28}$$

where the links K_+, K_-, and K_0 differ only at one crossing, as shown in Fig. 7. For definiteness we choose the shading shown in Fig. 7. Then the corresponding partition functions Z_+, Z_-, and Z_0 differ only by the factor arising at this crossing, which is respectively $u_+(n)$, $u_-(n)$, or 1, where $u_\pm(n)$ is given in (20). It is easy to show that the following identities hold for $N=2, 3$:

$$\begin{aligned} &N=2: &\quad u_+(n) + u_-(n) &= \sqrt{2}, &\quad n&=0, 1 \\ &N=3: &\quad e^{-\pi i/6} u_+(n) + e^{\pi i/6} u_-(n) &= 1, &\quad n&=0, 1, 2 \end{aligned} \tag{29}$$

These imply the same relations for Z_+, Z_-, and Z_0. Comparing with the Skein relation (28) for the Jones polynomial, we see that

$$\begin{aligned} I_K(2) &= (-1)^{c(K)+1}\, V_K(-i) \\ I_K(3) &= (-1)^{c(K)+1}\, V_K(e^{-\pi i/3}) \end{aligned} \tag{30}$$

where $c(K)$ is the number of components of the link K. The invariants (30) can further be related to the Homfly polynomial $P_K(t, z)$ by using the identity $(-1)^{c(K)+1}\, V_K(t) = P_K(-t, \sqrt{t} - 1/\sqrt{t})$.

As discussed in ref. 12, the invariants for higher values of N also satisfy certain Skein relations, but with higher-order crossings. For example, the invariant $I_K(4)$ satisfies

$$I_{K_{2+}}(4) - I_{K_+}(4) - iI_{K_-}(4) = -iI_{K_0}(4) \tag{31}$$

where $2+$ represents a crossing with two consecutive twists of the type u_+ shown in Fig. 7. Generally, the invariant $I_K(N)$ satisfies a Skein relation connecting $I_{K_0}(N)$, $I_{K_-}(N)$, $I_{K_+}(N)$,..., $I_{K_{[N/2]+}}(N)$, where $[N/2] = N/2$ for

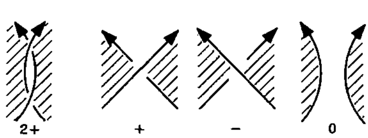

Fig. 7. Line configurations that can occur at a line crossing for writing down the Skein relation.

$N=$ even and $[N/2]=(N-1)/2$ for $N=$ odd, and $n+$ is a crossing of n consecutive twists of the type u_+. The Skein relation is obtained by writing out the identity

$$u_-(n) \prod_{p=0}^{[N/2]} [u_+(n) - u_+(p)] = 0, \qquad n=0, 1,..., N-1 \tag{32}$$

and making use of (2c). These relations, which are reminiscent of the Skein relations satisfied by the Akutsu–Wadati polynomials,[25,3] are not very useful for evaluating the invariant.

8. DISCUSSIONS

8.1. Mirror Image and Orientation Reversals

The invariant for the mirror image of a link is obtained by taking the complex conjugation. This follows from the fact that, in a mirror image, one interchanges $u_+ \leftrightarrow u_-$, $\bar{u}_+ \leftrightarrow \bar{u}_-$, and hence $\mathbf{M} \leftrightarrow -\mathbf{M}$, and by inspection of (20) and (23) one finds that these changes induce only a complex conjugation. Also, since $\mathbf{M}$ is independent of line orientations, from (23) we see that the reversal of the orientation of individual components in a link introduces only an overall factor $e^{i(N-1)\pi\Delta\zeta/4}$, where $\Delta\zeta$ is the induced change of $\zeta(K)$.

8.2. Invariants for Split Links

In standard notation, a link K is split if it can be deformed so that a hyperplane in $\mathbf{R}^3$ separates the link into two disjoint nonempty pieces, K_1 and K_2, say. By choosing the shading that leaves the infinite region unshaded as shown in Fig. 8, we see from a consideration of (3) that

$$I_K(N) = \sqrt{N}\, I_{K_1}(N)\, I_{K_2}(N) \tag{33}$$

where the factor of $\sqrt{N}$ comes from the changes of c_D. It follows that $I_K(N)$ has a factor $N^{(m-1)/2}$, where m is the number of disjoint pieces contained in the split link K.

Fig. 8. Consideration of split and connected links.

8.3. Invariants for Connected Links

The connected sum of two links K_1 and K_2, the link $K_1 \# K_2$, is obtained by cutting open both links and joining them as shown in Fig. 8. Denote by K the disjoint union of links K_1 and K_2 before they are connected. By considering (3) with the infinite region shaded and using (33), one finds

$$I_{K_1 \# K_2}(N) = N^{-1/2} I_K(N) = I_{K_1}(N)\, I_{K_2}(N) \tag{34}$$

Therefore, like the Jones and Homfly polynomials, $I_K(N)$ factorizes over connected sums of links.

8.4. Mutant Links

We next review the notion of mutant links[26] and show that $I_K(N)$ is unchanged under this operation. Suppose there is a simple closed curve in the plane which cuts a link K at four points only. By deforming the projection, one can place the interior of the curve inside a box as shown in Fig. 9a, with the orientations of the four incoming and outgoing lines as indicated. The part of the link inside the box is called a tangle. If the four lines are cut and reconnected after a half twist is put on the incoming lines, and the *opposite* half-twist on the outgoing lines as in Fig. 9b, we get a new link $\tilde{K}$ which is called a mutant of K.

Suppose now that we calculate $I_K(N)$ by using (3). We choose the shading so that spins are placed in the regions on the left and right sides of the tangle. The orientations of incoming and outgoing lines of the tangle

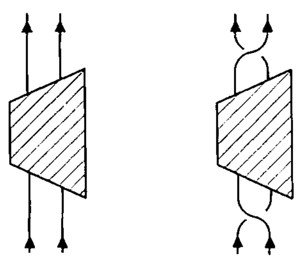

Fig. 9. (a) A tangle cut from a link K. (b) The tangle with two half twists added to its lines to be reconnected to K.

mean that these are different regions. The summand of the partition function for $\tilde{K}$ has an extra factor $u_+(a-b)\,u_-(a-b)$, where a and b are the spins in the left and right regions, respectively. By (2c), which comes from the Reidemeister move, this factor is 1. Hence the link K and its mutant $\tilde{K}$ have the same partition function and the same invariant.

Note that the above argument is quite general and establishes the general result that an invariant and its mutant are identical, provided that the invariant is derivable from an edge-interaction spin model. In particular, this applies to the Jones polynomial.

8.5. Links with Singular M

In writing down (25) we have assumed that the matrix $\mathbf{M}$ is nonsingular, i.e., $D \neq 0$. Indeed, we find this condition satisfied by all links with eight or fewer crossings except 8^3_{10} and 8^4_3. Now the expression $\mathbf{n} \cdot (\mathbf{Mn})$ in the exponent of the summand in (23) is that of a quadratic form over integral domains of the $S-1$ variables n_i, and the condition $D=0$ says that the quadratic form is singular. It can be shown (see, e.g., ref. 27) that such singular quadratic forms can always be written as regular (nonsingular) ones over fewer integral variables. After changing into these new variables, the summations in (23) over the l missing variables can be performed, yielding a factor $N^{l/2}$, where l is the degeneracy of the zero eigenvalue of $\mathbf{M}$, and the remaining summations can therefore be evaluated using the Gaussian summation formula.

This procedure can be illustrated in a special case as follows. When $D=0$ we know that the rows of the matrix $\mathbf{M}$ are not all linearly independent. This means that there exist real numbers c_i, $i=2, 3,..., S-1$, such that

$$M_{1j} = -\sum_{i=2}^{S-1} c_i M_{ij}, \qquad j = 1, 2, ..., S-1 \tag{35}$$

Since $\mathbf{M}$ is symmetric, we have also

$$M_{i1} = -\sum_{j=2}^{S-1} M_{ij} c_j, \qquad i = 1, 2,..., S-1 \tag{36}$$

and, after substituting (36) into (35) and setting $j=1$,

$$M_{11} = \sum_{i,j=2}^{S-1} c_i M_{ij} c_j \tag{37}$$

It follows that we have

$$
\begin{aligned}
\mathbf{n}\cdot(\mathbf{Mn}) &= \sum_{i,j=1}^{S-1} n_i M_{ij} n_j \\
&= n_1 M_{11} n_1 + \sum_{j=2}^{S-1} n_1 M_{1j} n_j + \sum_{i=2}^{S-1} n_i M_{1j} n_j + \sum_{i,j=2}^{S-1} n_i M_{ij} n_j \\
&= \sum_{i,j=2}^{S-1} (c_i n_1 - n_i)\, M_{ij} (c_j n_1 - n_j) \qquad (38)
\end{aligned}
$$

where we have used (35)–(37) in reaching the last step in (38).

Provided that all the c_i are integers, the expression (38) is now a quadratic form over $S-2$ integral variables, effectively crossing out the first row and first column of the matrix $\mathbf{M}$. If this reduced quadratic form is again singular, one repeats the process (again assuming integral c_i) until the resulting quadratic form is regular. The expression (23) can then be evaluated by applying the Gaussian summation identity.

APPENDIX A. THE GENERALIZED GAUSSIAN SUMMATION

In 1808 Gauss[28] obtained a remarkable summation identity, now known as the Gaussian summation formula, which reads

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{(2\pi i/N)n^2} = \frac{e^{i\pi/4}}{\sqrt{2}} (1 + e^{-i\pi N/2}) \qquad (A1)$$

The Gaussian summation identity (A1) can be generalized in a number of ways.[15,29,30] A simple generalization is the identity

$$\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\pi i M n^2/N + 2\pi i n y} = \frac{1}{\sqrt{M}} e^{i\pi/4} \sum_{m=0}^{M-1} e^{-\pi i N(m+y)^2/M} \qquad (A2)$$

valid for integral M, N, and $N(2y+M)=$ an even integer, which recovers (A1) upon taking $M=2$ and $y=0$. A multidimensional summation generalization which we use in arriving at (25) is the following.

Let $\mathbf{M}$ be a nonsingular $p\times p$ symmetric matrix with integer entries (positive or negative). We denote by $\mathbf{n}=(n_1,\dots,n_p)$, where the n_i are integers, a vector in $\mathbf{Z}^p$, and by $\mathbf{Mn}$ the vector with components $\sum_{j=1}^{p} M_{ij} n_j$. The collection of vectors $\{\mathbf{Mn} : \mathbf{n}\in\mathbf{Z}^p\}$ forms a p-dimensional sublattice of $\mathbf{Z}^p$. Let Δ be a fundamental domain (unit cell) of this sublattice. Similarly we define $x = (x_1,..., x_p)$ where the x_i are real, a vector in the p-dimensional space $\mathbf{R}^p$. For all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^p$, define

$$u(\mathbf{x}, \mathbf{y}) = \frac{1}{N^{p/2}} \exp\left[\frac{\pi i}{N} \mathbf{x} \cdot (\mathbf{M}\mathbf{x}) + 2\pi i \mathbf{x} \cdot \mathbf{y}\right]$$
$$v(\mathbf{x}) = \frac{1}{\sqrt{D}} \exp[-\pi i N \mathbf{x} \cdot (\mathbf{M}^{-1}\mathbf{x})] \tag{A3}$$

where $D = |\det \mathbf{M}|$. Let $\eta(\mathbf{M})$ denote the signature of $\mathbf{M}$, which is the number of positive eigenvalues minus the number of negative eigenvalues of $\mathbf{M}$. Also let C_N be the set of N^p discrete points $C_N = \{\mathbf{n} \in \mathbf{Z}^p : 0 \leqslant n_i \leqslant N-1, i = 1,..., p\}$.

For $\mathbf{y} \in \mathbf{R}^p$ satisfying the condition

$$N(2y_i + M_{ii}) = \text{even integer}, \qquad i = 1,..., p \tag{A4}$$

we have the *generalized Gaussian summation* formula

$$\sum_{\mathbf{n} \in C_N} u(\mathbf{n}, \mathbf{y}) = e^{i\pi\eta(\mathbf{M})/4} \sum_{\mathbf{m} \in \Delta} v(\mathbf{m} + \mathbf{y}) \tag{A5}$$

For $p = 1$, (A5) reduces to (A2).

APPENDIX B. TABLE OF INVARIANTS FOR LINKS WITH EIGHT OR FEWER CROSSINGS

The invariant for a link K is generally given by the expression

$$I_K(N) = \left(\frac{N^{l/2}}{\sqrt{D_1}}\right) e^{ik\pi/4} \sum_{n=0}^{2D_2-1} c(n)\, e^{-in\pi N/D_2} \tag{B1}$$

where l is the degeneracy of the zero eigenvalue of $\mathbf{M}$, which is equal to zero unless $D = 0$ as in 8^3_{10} and 8^4_3. The following table gives the values of $[k, D_1, D_2]\{c(n)_n\}$ with nonzero $c(n)$ listed. For example, the invariant for the link 7_1 is

$$[2, 7, 7]\{1_0, 2_2, 2_4, 2_8\} \rightarrow \frac{1}{\sqrt{7}} e^{i2\pi/4}(1 + 2e^{-i2\pi N/7} + 2e^{-i4\pi N/7} + 2e^{-i8\pi N/7})$$

The table also lists $D = |\det \mathbf{M}|$ and $w(K) = n_+ - n_-$, the number of + crossings minus the number of − crossings, to specify the direction of line orientations. Generally, $D = \sum_{n=0}^{2D_2-1} c(n)$ $(l=0)$, D_1 and D_2 are factors of D, and, in a few cases, $D_2 = 2D$.

K	D	$w(K)$	$I_K(N)$
3_1	3	+3	$[-2, 3, 3]\{1_0, 2_4\}$
4_1	5	0	$[0, 5, 5]\{1_0, 2_2, 2_8\}$
5_1	5	+5	$[4, 5, 5]\{1_0, 2_4, 2_6\}$
5_2	7	+5	$[-2, 7, 7]\{1_0, 2_6, 2_{10}, 2_{12}\}$
6_1	9	+2	$[0, 9, 9]\{3_0, 2_4, 2_{10}, 2_{16}\}$
6_2	11	+2	$[-2, 11, 11]\{1_0, 2_4, 2_{12}, 2_{14}, 2_{16}, 2_{20}\}$
6_3	13	0	$[0, 13, 13]\{1_0, 2_2, 2_6, 2_8, 2_{18}, 2_{20}, 2_{24}\}$
2_1^2	2	+2	$[1, 2, 2]\{1_0, 1_1\}$
4_1^2	4	−4	$[3, 4, 4]\{1_0, 2_3, 1_4\}$
		+4	$[-1, 4, 4]\{1_0, 1_4, 2_7\}$
5_1^2	8	+1	$[-1, 2, 8]\{1_0, 1_3, 1_{11}, 1_{12}\}$
6_1^2	6	−6	$[-3, 6, 6]\{1_0, 2_5, 2_8, 1_9\}$
		+6	$[-1, 6, 6]\{1_0, 1_3, 2_8, 2_{11}\}$
6_2^2	10	+6	$[-3, 10, 10]\{1_0, 1_5, 2_8, 2_{12}, 2_{13}, 2_{17}\}$
6_3^2	12	−6	$[3, 12, 12]\{1_0, 2_4, 4_7, 1_{12}, 2_{15}, 2_{16}\}$
		+2	$[-1, 12, 12]\{1_0, 2_3, 2_4, 1_{12}, 2_{16}, 4_{19}\}$
6_1^3	12	−6	$[2, 12, 12]\{1_0, 6_4, 3_{12}, 2_{16}\}$
		+2	$[-2, 12, 12]\{3_0, 2_4, 1_{12}, 6_{16}\}$
6_2^3	16	0	$[0, 4, 4]\{2_0, 3_2, 3_6\}$
6_3^3	4	−2	$[0, 4, 4]\{3_0, 1_4\}$
		+6	$[4, 4, 4]\{1_0, 3_4\}$
7_1	7	+7	$[2, 7, 7]\{1_0, 2_2, 2_4, 2_8\}$
7_2	11	+7	$[-2, 11, 11]\{1_0, 2_4, 2_{12}, 2_{14}, 2_{16}, 2_{20}\}$
7_3	13	+7	$[4, 13, 13]\{1_0, 2_4, 2_{10}, 2_{12}, 2_{14}, 2_{16}, 2_{22}\}$
7_4	15	+7	$[-2, 15, 15]\{1_0, 2_6, 4_{14}, 2_{20}, 2_{24}, 4_{26}\}$
7_5	17	+7	$[4, 17, 17]\{1_0, 2_6, 2_{10}, 2_{12}, 2_{14}, 2_{20}, 2_{22}, 2_{24}, 2_{28}\}$
7_6	19	+3	$[-2, 19, 19]\{1_0, 2_4, 2_6, 2_{16}, 2_{20}, 2_{24}, 2_{26}, 2_{28}, 2_{30}, 2_{36}\}$
7_7	21	+1	$[0, 21, 21]\{1_0, 2_6, 4_{10}, 2_{12}, 2_{24}, 2_{28}, 4_{34}, 4_{40}\}$
7_1^2	14	−3	$[3, 14, 14]\{1_0, 2_4, 1_7, 2_8, 2_{11}, 2_{15}, 2_{16}, 2_{23}\}$
		+1	$[1, 14, 14]\{1_0, 2_1, 2_4, 2_8, 2_9, 2_{16}, 1_{21}, 2_{25}\}$
7_2^2	18	−3	$[1, 18, 18]\{3_0, 2_5, 2_8, 3_9, 2_{17}, 2_{20}, 2_{29}, 2_{32}\}$
		+1	$[0, 18, 36]\{2_1, 2_7, 3_9, 2_{25}, 2_{31}, 2_{49}, 2_{55}, 3_{63}\}$
7_3^2	16	+3	$[-1, 4, 16]\{1_0, 1_7, 1_{15}, 1_{16}, 1_{23}, 2_{28}, 1_{31}\}$
7_4^2	16	+3	$[-3, 4, 16]\{1_0, 1_5, 1_{13}, 1_{16}, 2_{20}, 1_{21}, 1_{29}\}$
7_5^2	20	−7	$[3, 20, 20]\{1_0, 4_7, 2_8, 2_{12}, 2_{15}, 1_{20}, 4_{23}, 2_{28}, 2_{32}\}$
		+1	$[-1, 20, 20]\{1_0, 4_3, 2_8, 2_{12}, 1_{20}, 4_{27}, 2_{28}, 2_{32}, 2_{35}\}$

K	D	$w(K)$	$I_K(N)$
7_6^2	24	+1	$[-1, 6, 24]\{1_0, 1_3, 2_{11}, 1_{12}, 1_{27}, 2_{32}, 2_{35}, 2_{44}\}$
7_7^2	4	−1	$[-1, 4, 4]\{1_0, 1_4, 2_7\}$
		+7	$[-3, 4, 4]\{1_2, 2_5, 1_6\}$
7_8^2	8	+3	$[-1, 2, 8]\{1_0, 1_7, 1_{12}, 1_{15}\}$
7_1^3	20	+1	$[0, 20, 5]\{3_0, 6_2, 2_3, 1_5, 2_7, 6_8\}$
8_1	13	+4	$[0, 13, 13]\{1_0, 2_2, 2_6, 2_8, 2_{18}, 2_{20}, 2_{24}\}$
8_2	17	+4	$[4, 17, 17]\{1_0, 2_6, 2_{10}, 2_{12}, 2_{14}, 2_{20}, 2_{22}, 2_{24}, 2_{28}\}$
8_3	17	0	$[0, 17, 17]\{1_0, 2_2, 2_4, 2_8, 2_{16}, 2_{18}, 2_{26}, 2_{30}, 2_{32}\}$
8_4	19	0	$[2, 19, 19]\{1_0, 2_2, 2_8, 2_{10}, 2_{12}, 2_{14}, 2_{18}, 2_{22}, 2_{32}, 2_{34}\}$
8_5	21	+4	$[4, 21, 21]\{1_0, 2_6, 2_{12}, 2_{14}, 4_{20}, 2_{24}, 4_{26}, 4_{38}\}$
8_6	23	+4	$[-2, 23, 23]\{1_0, 2_{10}, 2_{14}, 2_{20}, 2_{22}, 2_{28}, 2_{30}, 2_{34}, 2_{38}, 2_{40}, 2_{42}, 2_{44}\}$
8_7	23	+2	$[-2, 23, 23]\{1_0, 2_{10}, 2_{14}, 2_{20}, 2_{22}, 2_{28}, 2_{30}, 2_{34}, 2_{38}, 2_{40}, 2_{42}, 2_{44}\}$
8_8	25	+2	$[2, 25, 50]\{2_{13}, 2_{17}, 5_{25}, 2_{33}, 2_{37}, 2_{53}, 2_{57}, 2_{73}, 2_{77}, 2_{93}, 2_{97}\}$
8_9	25	0	$[0, 25, 25]\{5_0, 2_2, 2_8, 2_{12}, 2_{18}, 2_{22}, 2_{28}, 2_{32}, 2_{38}, 2_{42}, 2_{48}\}$
8_{10}	27	+2	$[-2, 27, 27]\{3_0, 2_4, 2_{10}, 2_{16}, 2_{22}, 2_{28}, 2_{34}, 6_{36}, 2_{40}, 2_{46}, 2_{52}\}$
8_{11}	27	+4	$[-2, 27, 27]\{3_0, 2_4, 2_{10}, 2_{16}, 2_{22}, 2_{28}, 2_{34}, 6_{36}, 2_{40}, 2_{46}, 2_{52}\}$
8_{12}	29	0	$[0, 29, 29]\{1_0, 2_2, 2_8, 2_{10}, 2_{12}, 2_{14}, 2_{18}, 2_{26}, 2_{32}, 2_{40}, 2_{44}, 2_{46}, 2_{48}, 2_{50}, 2_{56}\}$
8_{13}	29	+2	$[0, 29, 29]\{1_0, 2_2, 2_8, 2_{10}, 2_{12}, 2_{14}, 2_{18}, 2_{26}, 2_{32}, 2_{40}, 2_{44}, 2_{46}, 2_{48}, 2_{50}, 2_{56}\}$
8_{14}	31	+4	$[-2, 31, 31]\{1_0, 2_6, 2_{12}, 2_{22}, 2_{24}, 2_{26}, 2_{30}, 2_{34}, 2_{42}, 2_{44}, 2_{46}, 2_{48}, 2_{52}, 2_{54}, 2_{58}, 2_{60}\}$
8_{15}	33	+8	$[4, 33, 33]\{1_0, 2_6, 4_{10}, 2_{18}, 2_{22}, 2_{24}, 4_{28}, 2_{30}, 4_{40}, 4_{46}, 4_{52}, 2_{54}\}$
8_{16}	35	+2	$[-2, 35, 35]\{1_0, 4_4, 2_{14}, 4_{16}, 2_{30}, 4_{36}, 4_{44}, 4_{46}, 2_{50}, 2_{56}, 2_{60}, 4_{64}\}$
8_{17}	37	0	$[0, 37, 37]\{1_0, 2_2, 2_6, 2_8, 2_{14}, 2_{18}, 2_{20}, 2_{22}, 2_{24}, 2_{32}, 2_{42}, 2_{50}, 2_{52}, 2_{54}, 2_{56}, 2_{60}, 2_{66}, 2_{68}, 2_{72}\}$
8_{18}	45	0	$[0, 45, 15]\{1_0, 8_2, 8_8, 4_{10}, 2_{12}, 2_{18}, 4_{20}, 8_{22}, 8_{28}\}$
8_{19}	3	+8	$[2, 3, 3]\{1_0, 2_2\}$
8_{20}	9	+2	$[0, 9, 9]\{3_0, 2_2, 2_8, 2_{14}\}$
8_{21}	15	+4	$[-2, 15, 15]\{1_0, 2_{10}, 2_{12}, 2_{18}, 4_{22}, 4_{28}\}$
8_1^2	8	+8	$[1, 2, 8]\{1_0, 1_1, 1_4, 1_9\}$
8_2^2	16	+8	$[3, 4, 16]\{1_0, 1_3, 1_{11}, 2_{12}, 1_{16}, 1_{19}, 1_{27}\}$

822/78/5-6-6

K	D	$w(K)$	$I_K(N)$
8_3^2	22	-4	$[1, 22, 22]\{1_0, 2_1, 2_4, 2_5, 2_9, 2_{12}, 2_{16}, 2_{20}, 2_{25}, 1_{33}, 2_{36}, 2_{37}\}$
		$+8$	$[3, 22, 22]\{1_0, 2_3, 2_4, 1_{11}, 2_{12}, 2_{15}, 2_{16}, 2_{20}, 2_{23}, 2_{27}, 2_{31}, 2_{36}\}$
8_4^2	24	-4	$[1, 6, 24]\{1_0, 1_3, 2_4, 2_{16}, 2_{19}, 1_{27}, 1_{39}, 2_{43}\}$
		$+8$	$[3, 6, 24]\{1_0, 2_7, 1_{12}, 1_{15}, 2_{16}, 2_{28}, 2_{31}, 1_{39}\}$
8_5^2	26	$+4$	$[-3, 26, 26]\{1_0, 2_5, 2_8, 1_{13}, 2_{20}, 2_{21}, 2_{24}, 2_{28}, 2_{32}, 2_{33}, 2_{37}, 2_{41}, 2_{44}, 2_{45}\}$
8_6^2	20	0	$[1, 20, 20]\{1_0, 4_1, 2_4, 4_9, 2_{16}, 1_{20}, 2_{24}, 2_{25}, 2_{26}\}$
		$+8$	$[-3, 20, 20]\{1_0, 2_4, 2_5, 2_{16}, 1_{20}, 4_{21}, 2_{24}, 4_{29}, 2_{36}\}$
8_7^2	30	0	$[-1, 30, 30]\{1_0, 4_{11}, 1_{15}, 2_{20}, 2_{24}, 2_{35}, 2_{36}, 2_{39}, 4_{44}, 2_{51}, 4_{56}, 4_{59}\}$
		$+4$	$[-3, 30, 30]\{1_0, 2_5, 2_9, 2_{20}, 2_{21}, 2_{24}, 4_{29}, 2_{36}, 4_{41}, 4_{44}, 1_{45}, 4_{56}\}$
8_8^2	34	$+2$	$[-1, 34, 34]\{1_0, 2_4, 2_8, 2_{15}, 2_{16}, 2_{19}, 2_{32}, 2_{35}, 2_{36}, 2_{43}, 2_{47}, 1_{51}, 2_{52}, 2_{55}, 2_{59}, 2_{60}, 2_{64}, 2_{67}\}$
8_9^2	28	-4	$[1, 28, 28]\{1_0, 4_1, 2_4, 2_8, 4_9, 2_{16}, 4_{25}, 1_{28}, 2_{32}, 2_{36}, 2_{44}, 2_{49}\}$
		0	$[-1, 28, 28]\{2_2, 4_{11}, 1_{14}, 2_{18}, 2_{22}, 2_{30}, 2_{35}, 1_{42}, 4_{43}, 2_{46}, 2_{50}, 4_{51}\}$
8_{10}^2	32	0	$[-1, 8, 32]\{2_0, 1_3, 1_{11}, 2_{12}, 1_{19}, 1_{27}, 1_{35}, 1_{43}, 2_{44}, 2_{48}, 1_{51}, 1_{59}\}$
8_{11}^2	28	0	$[-1, 28, 28]\{1_0, 2_4, 2_8, 4_{11}, 2_{16}, 1_{28}, 2_{32}, 2_{35}, 2_{36}, 4_{43}, 2_{44}, 4_{51}\}$
		$+8$	$[3, 28, 28]\{1_0, 2_4, 2_7, 2_8, 4_{15}, 2_{16}, 4_{23}, 1_{28}, 2_{32}, 2_{36}, 4_{39}, 2_{44}\}$
8_{12}^2	32	$+2$	$[1, 8, 32]\{2_0, 1_3, 1_{11}, 2_{12}, 1_{19}, 1_{27}, 1_{35}, 1_{43}, 2_{44}, 2_{48}, 1_{51}, 1_{59}\}$
8_{13}^2	40	$+2$	$[-1, 10, 40]\{1_0, 2_{11}, 2_{16}, 2_{19}, 1_{35}, 2_{44}, 2_{51}, 2_{59}, 1_{60}, 2_{64}, 1_{75}, 2_{76}\}$
8_{14}^2	36	0	$[1, 36, 12]\{1_0, 4_4, 8_7, 4_8, 1_{12}, 2_{15}, 4_{16}, 4_{20}, 8_{23}\}$
		$+8$	$[-3, 36, 12]\{1_0, 2_3, 4_4, 4_8, 8_{11}, 1_{12}, 4_{16}, 8_{19}, 4_{20}\}$
8_{15}^2	8	0	$[-1, 2, 8]\{1_0, 1_7, 1_{12}, 1_{15}\}$
8_{16}^2	12	-4	$[1, 12, 12]\{1_0, 4_1, 2_4, 2_9, 1_{12}, 2_{16}\}$
		$+4$	$[-1, 12, 12]\{2_3, 1_6, 2_{10}, 1_{18}, 4_{19}, 2_{22}\}$
8_1^3	20	-8	$[2, 20, 10]\{1_0, 2_2, 4_3, 4_7, 2_8, 1_{10}, 2_{12}, 2_{15}, 2_{18}\}$
		0	$[-2, 20, 10]\{1_0, 2_2, 2_5, 2_8, 1_{10}, 2_{12}, 4_{13}, 4_{17}, 2_{18}\}$
		$+4$	$[4, 20, 10]\{2_0, 2_3, 1_5, 2_7, 4_8, 4_{12}, 2_{13}, 1_{15}, 2_{17}\}$

K	D	$w(K)$	$I_K(N)$
8_2^3	28	-8	$[4, 28, 14]\{1_0, 4_5, 2_6, 2_{10}, 2_{12}, 4_{13}, 1_{14}, 4_{17}, 2_{20}, 2_{21}, 2_{24}, 2_{26}\}$
		0	$[0, 28, 14]\{1_0, 4_3, 2_6, 2_7, 2_{10}, 2_{12}, 1_{14}, 4_{19}, 2_{20}\ 2_{24}, 2_{26}, 4_{27}\}$
		$+2$	$[-1, 28, 28]\{2_5, 2_7, 2_{13}, 2_{17}, 1_{21}, 4_{31}, 2_{33}, 2_{41}, 2_{45}, 4_{47}, 1_{49}, 4_{55}\}$
8_3^3	28	-4	$[2, 28, 7]\{1_0, 6_3, 6_5, 2_6, 3_7, 2_{10}, 2_{12}, 6_{13}\}$
		$+4$	$[-2, 28, 7]\{3_0, 2_3, 2_5, 6_6, 1_7, 6_{10}, 6_{12}, 2_{13}\}$
		$+6$	$[-3, 28, 28]\{2_5, 2_{13}, 6_{17}, 1_{21}, 6_{33}, 6_{41}, 2_{45}, 3_{49}\}$
8_4^3	32	0	$[0, 8, 8]\{1_0, 2_2, 2_3, 1_4, 2_5, 1_8, 2_{11}, 1_{12}, 2_{13}, 2_{14}\}$
		$+8$	$[4, 8, 8]\{1_0, 2_3, 1_4, 2_5, 2_6, 1_8, 2_{10}, 2_{11}, 1_{12}, 2_{13}\}$
8_5^3	32	-1	$[2, 8, 16]\{1_0, 2_1, 2_4, 1_8, 2_9, 2_{12}, 1_{16}, 2_{17}, 1_{24}, 2_{25}\}$
		$+2$	$[0, 8, 16]\{1_0, 2_1, 2_4, 1_8, 2_9, 1_{16}, 2_{17}, 1_{24}, 2_{25}, 2_{28}\}$
		$+4$	$[-1, 8, 16]\{2_0, 1_4, 2_5, 1_{12}, 2_{13}, 1_{20}, 2_{21}, 2_{24}, 1_{28}, 2_{29}\}$
8_6^3	36	-4	$[2, 36, 6]\{5_0, 2_2, 10_3, 2_4, 5_6, 4_7, 2_8, 2_{10}, 4_{11}\}$
		0	$[0, 36, 6]\{10_0, 2_1, 5_3, 4_4, 2_5, 2_7, 4_8, 5_9, 2_{11}\}$
		$+4$	$[-2, 36, 18]\{5_0, 4_3, 2_6, 2_{12}, 4_{15}, 5_{18}, 2_{24}, 10_{27}, 2_{30}\}$
8_7^3	4	-4	$[0, 4, 2]\{2_0, 1_1, 1_3\}$
		0	$[-2, 4, 2]\{1_0, 1_2, 2_3\}$
		$+8$	$[2, 4, 2]\{1_0, 2_1, 1_2\}$
8_8^3	12	-4	$[2, 12, 6]\{2_0, 2_1, 1_3, 4_4, 2_7, 1_9\}$
		0	$[0, 12, 6]\{1_0, 4_1, 2_4, 1_6, 2_9, 2_{10}\}$
		$+8$	$[4, 12, 6]\{1_0, 2_3, 2_4, 1_6, 4_7, 2_{10}\}$
8_9^3	16	-4	$[2, 4, 4]\{1_0, 2_1, 2_2, 1_4, 2_5\}$
		0	$[0, 4, 4]\{2_0, 1_2, 2_3, 1_6, 2_7\}$
		$+4$	$[-2, 4, 4]\{1_0, 2_1, 1_4, 2_5, 2_6\}$
8_{10}^3	0	-4	$[3, 4, 4]\{1_2, 2_3, 1_6\}$, $l=1$
		0	$[1, 4, 4]\{1_0, 2_1, 1_4\}$, $l=1$
		$+4$	$[-3, 4, 4]\{1_0, 1_4, 2_5\}$, $l=1$
8_1^4	32	-8	$[3, 8, 8]\{1_0, 4_3, 3_4, 3_8, 4_{11}, 1_{12}\}$
		-4	$[1, 8, 8]\{3_0, 3_4, 4_7, 1_8, 1_{12}, 4_{15}\}$
		0	$[-1, 8, 8]\{3_0, 4_3, 1_4, 1_8, 4_{11}, 3_{12}\}$
8_2^4	16	-4	$[1, 4, 4]\{2_0, 3_1, 2_4, 1_5\}$
		-4	$[-1, 4, 4]\{2_2, 1_3, 2_6, 3_7\}$
		$+4$	$[-3, 4, 4]\{2_0, 1_1, 2_4, 3_5\}$
8_3^4	0	0	$[0, 4, 1]\{3_0, 1_1\}$, $l=1$
		$+8$	$[4, 4, 1]\{1_0, 3_1\}$, $l=1$

ACKNOWLEDGMENTS

We would like to thank Helen Au-Yang for pointing out the resemblance of properties of the chiral Potts model weights (12) with those of the Reidemeister moves (2), which has led to this investigation. We also thank Alex Suciu for helpful discussions. Work by FYW and PP has been supported in part by National Science Foundation grants DMR-9313648, INT-9207261, and a Northeastern University Research and Scholarship Development Fund grant. Work by CK has been supported in part by a Northeastern University Research award.

REFERENCES

1. V. F. R. Jones, *Pacific J. Math.* **137**:311 (1989).
2. M. Wadati, T. Deguchi, and Y. Akutsu, *Phys. Rep.* **180**:247 (1989).
3. F. Y. Wu, *Rev. Mod. Phys.* **64**:1099 (1992).
4. V. F. R. Jones, *Bull. Am. Math. Soc.* **12**:103 (1985).
5. P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. C. Millett, and A. Ocneanu, *Bull. Am. Math. Soc.* **12**:239 (1985).
6. D. Goldschmidt and V. F. R. Jones, *Geom. Dedicata* **31**:165 (1989); V. F. R. Jones, *Commun. Math. Phys.* **125**:459 (1989).
7. E. Date, M. Jimbo, K. Miki, and T. Miwa, *Pacific J. Math.* **154**:37 (1992).
8. F. Jaeger, *Geom. Dedicata* **44**:23 (1992).
9. P. de la Harpe and V. F. R. Jones, *J. Comb. Theory B* **57**:207 (1993).
10. E. Bannai and E. Bannai, *Mem. Fac. Sci. Kyushu Univ. A* **47**:397 (1993).
11. P. de la Harpe, *Pacific J. Math.* **162**:57 (1994).
12. F. Y. Wu, P. Pant, and C. King, *Phys. Rev. Lett.* **72**:3937 (1994).
13. H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. L. Yan, *Phys. Lett.* **123A**:219 (1987).
14. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, *Phys. Lett. A* **128**:138 (1988).
15. C. L. Siegel, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Klasse* **1**:1 (1960).
16. T. Kobayashi, H. Murakami and J. Murakami, *Proc. Japan Acad.* **64A**:235 (1988).
17. R. J. Baxter, S. B. Kelland, and F. Y. Wu, *J. Phys. A* **9**:397 (1976).
18. K. Reidemeister, *Knotentheorie* (Chelsea, New York, 1948).
19. F. Y. Wu and Y. K. Wang, *J. Math. Phys.* **17**:439 (1976).
20. V. A. Fateev and A. B. Zamolodchikov, *Phys. Lett. A* **92**:37 (1982).
21. F. Harary, *Graph Theory* (Addison-Wesley, New York, 1971).
22. L. Goeritz, *Math. Z* **36**:647 (1933).
23. G. Burde and H. Zieschang, *Knots* (Walter de Gruyter, New York, 1985).
24. L. A. Traldi, *Math. Z.* **188**:203 (1985).
25. Y. Akutsu and M. Wadati, *J. Phys. Soc. Japan* **56**:3039 (1987).
26. W. B. R. Lickorish and K. C. Millett, *Math. Mag.* **61**:3 (1988).
27. J. W. S. Cassels, *Rational Quadratic Forms* (Academic Press, London, 1972).
28. C. F. Gauss, *Summatio quarundam serierum singularium, 1808*, in *Werke II* (Göttingen, 1870).
29. A. Krazer, in *Festschrift H. Weber* (Leipzig u., Berlin, 1912).
30. M. Eichler, *Quadratische Formen und Orthogonale Gruppen* (Springer-Verlag, Berlin, 1952).